W4L2 - CALCULATING THE LAPLACE TRANSFORM

Def: Suppose $f(t)$ is a function of t. The Laplace Transform of f is the following function of s:

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt \quad , \text{ for } s>0$$

Compute the Laplace Transform of $f(t)=t$

$$F(s) = \int_0^\infty te^{-st}\,dt \quad \leftarrow \text{Improper Integral}$$

$$\lim_{w\to\infty} \int_0^w te^{-st}\,dt$$

$$\lim_{w\to\infty} \left[t\left(\frac{e^{-st}}{-s}\right) - \frac{1}{s^2}\left(e^{-st}\right)\right]\Bigg|_{0=t}^{w=t}$$

$$\lim_{w\to\infty} \left[w\left(\frac{e^{-sw}}{-s}\right) - \frac{1}{s^2}\left(e^{-sw}\right) - \left(0 - \frac{1}{s^2}\cdot 1\right)\right]$$

$$\lim_{w\to\infty} \left[\frac{-1}{s}\left(\cancel{we^{-sw}}\right) - \frac{1}{s^2}\cancel{e^{-sw}} + \frac{1}{s^2}\right]$$

$\infty \quad 0$

$e^{-s\infty} \to e^{-\infty} \to 0$

$$\lim_{w\to\infty} \frac{w}{e^{sw}} \overset{L}{=} \lim_{w\to\infty} \frac{1}{e^{sw}\cdot s} = \frac{1}{e^{\infty}\cdot s} = 0$$

$$= \boxed{\frac{1}{s^2}}$$

Aside

$$\int te^{-st}\,dt$$

IBP: $\int u\,dv = uv - \int v\,du$

$u = t \qquad v = \frac{e^{-st}}{-s}$

$du = dt \qquad dv = e^{-st}\,dt$

$$= t\left(\frac{e^{-st}}{-s}\right) - \int \frac{e^{-st}}{-s}\,dt$$

$$= t\left(\frac{e^{-st}}{-s}\right) + \frac{1}{s}\int e^{-st}\,dt$$

$$= t\left(\frac{e^{-st}}{-s}\right) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right) + C$$